Classical Solutions of 2D String Theory in any Curved Spacetime[*]

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Abstract

The complete set of solutions of two dimensional classical string theory are constructed for any curved spacetime. They describe folded strings moving in curved spacetime. Surprizing stringy behavior becomes evident at singularities such as black holes. The solutions are given in the form of a map from the world sheet to target spacetime, where the world sheet has to be divided into lattice-like patches corresponding to different maps. A recursion relation analogous to a "transfer matrix" that connects these maps into a single continuous map is derived. This "transfer matrix" encodes the properties of the world sheet lattice on the one hand and the geometry of spacetime on the other hand. The solutions are completely classified by their behavior in the asymptotically flat region of spacetime where they reduce, as boundary conditions, to the folded string solutions that have been known for 19 years.

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1 Introduction

The original physical motivations for studying string theory were: (1) understanding unification of forces including quantum gravity, and (2) understanding the Standard Model. In recent years it has become more and more evident that these goals should be examined in the presence of curved 4D space-time string backgrounds. The construction of 4D curved space-time string theories that correspond to exact conformal theories have provided models in which various questions can be investigated[1].

The usual scenario of flat 4D plus extra curved dimensions may not be the right approach for making predictions about the Standard Model. I believe that the gauge symmetries and spectrum of quark + lepton families, which are the main ingredients of the Standard Model, were probably fixed during the early times in the evolution of the Universe. At such times 4D space-time was curved. Since curvature contributes to the central charge and other topological aspects of String Theory, it is likely that the predictions of String Theory under such conditions may be quite different than the flat 4D approach. Therefore, I believe that String Theory in curved space-time must be better understood before attempting to make connections to low energy physics. One should consider all kinds of curved backgrounds, not only the traditional cosmological backgrounds, since the passage from curved space-time to flat space-time may involve various phase transitions, including inflation of a small region of the original curved universe to today's universe that is essentially homogeneous and flat. The gauge bosons, and chiral families of quarks and leptons in a small region of the early curved universe would become the ones observed in today's inflated flat universe. The possibility of such a scenario suggests that curved space-time string theory deserves intensive study. In addition, the issues surrounding gravitational singularities should be answered in the context of curved space-time string theory, as it is the only known theory of quantum gravity.

In this paper we are interested in two aspects of string theory: (i) strings in curved space-time and (ii) folded strings. They are both explored simultaneously in the complete classical solution of 2D string theory that we present here. First, we feel it is important to understand classical string theory

in curved space-time in order to understand and interpret quantum string theory in curved space-time. This is relevant to fundamental questions of singularities in gravitational physics, as well as stringy questions about the early universe and its influence on the low energy spectrum of quarks and leptons. In 2D the only non-trivial stringy solutions turn out to be necessarily folded strings, and therefore they are the only path toward analyzing such questions in a toy model. Second, there has been a long-standing interest in exploring consistent generalizations of non-critical strings with the hope that they may be relevant for some branch of physics. Folded strings fall into this category, especially in the area of string-QCD relations.

Two dimensional string theory in *flat space-time* was discussed in 1975 by BBHP [2] in the conformal, lightcone, and temporal gauges. The classical folded string solutions were obtained, a semiclassical quantization was performed, and agreement of results in various gauges was displayed. Furthermore, the Lorentz covariance and consistency of the quantum theory was proven by showing the closure of the 2D Poincaré algebra, for which the proof was clearest in the lightcone gauge [3]. BBHP discovered that the non-trivial classical motions correspond to longitudinal oscillations of folded strings. Folds as well as end points (if the string is open) move at the speed of light and oscillate against each other. These massless points present an anomaly that needs to be treated carefully. BBHP showed that by making these points massive, analyzing the motion, and sending the mass to zero at the end, the physics could be understood most satisfactorily. However, the mathematics is simplest in the conformal gauge in the massless limit, where the same classical motions are recovered provided one is careful [2].

The conformal gauge approach was recently generalized to curved space-time. The general *classical folded string solutions* were obtained for any 2D curved space-time and the motion in the 2D black hole was physically interpreted [4]. It was shown that the admissible solutions are those that smoothly connect to the folded string solutions in flat space-time, since far away from singularities curved space-time approaches flat space-time. Furthermore, by regarding curved space-time as a continuous deformation of flat space-time, except for singularities, one can intuitively guess the general behavior of the string motion as being similar to the folded string solutions. Therefore, away from singularities, the minimal 2D target space surface swept by the string turns out to be similar to the one in flat space-time, but its detailed shape obeys certain global conservation rules dictated by the curved metric and an

3

associated "transfer matrix". In the vicinity of singularities the same conservation laws are obeyed and they lead to certain surprises in the motion of strings. Using these solutions the swallowing of a string by a black hole was discussed, showing that new unusual stringy features emerge, such as the tunneling of the string into the region beyond the black hole (the bare singularity region) that is forbidden for particle geodesics, as well as other new effects.

To avoid confusion it is useful to emphasize that the folded string states are properties of the 2D quantum string theory as well. In a covariant quantization, they exist in addition to the "special momentum states" of the 2D quantum string theory that have been discussed in recent years. As pointed out many times in our past work, folded 2D-string states are present in the $d = 2$ and $c \leq 25$ sector of the quantum theory. In simple string models, when it has been possible to compute the spectrum, their norm is positive and is proportional to $(c - 26)$. Only if $d = 2$ and $c = 26$ simultaneously (e.g. $d = 2$ flat space-time with linear dilaton such that $c = 26$) the folded string states become zero norm states and then the special discrete momentum states survive as the only stringy states. A simple model in which these properties may be easily seen is the *covariant* quantization of the 2D string theory, in which the physical states are identified as the subset that satisfies the Virasoro constraints. For example, it has been known for a long time that the $d \leq 25$ sector of the flat string theory has non-trivial positive norm states (including for $d = 2$) that satisfy the Virasoro constraints and that there are no ghosts [5].

The possibility of a more general *interacting* string theory that includes folded strings, and the probable close connection with large-N QCD, provides additional motivation for studying folded strings. For example, it has been expected for a long time that there is an interacting string version of QCD in 2D to 4D. Some relations that were discovered a long time ago can be re-examined with a new point of view and generalized to curved space-time. There are relations that involve zero fold strings as well as folded strings:

- The open, *zero fold* string, with spinors attached at the ends [6], and with interactions at the end points [7], was shown to reproduce the low orders of the perturbation series of 2D large-N QCD with quarks [7], in flat space-time. This equivalence includes the 't Hooft spectrum [8], the 1/N strong interactions within QCD [9], and the 1/N Electro-weak

interactions with external fields [10]. These features were reproduced within string theory by a many-body type action describing propagation, string-string interactions at end points, and interactions with external fields [7].

- More recently it has become apparent [11] that folded strings in flat 2D space-time are closely connected to the flat space-time large-N 2D gauge theory interacting with fermions or bosons in the adjoint representation [12] [13]. The QCD flux (i.e. string) folds at the location of the adjoint fermion or boson, thus associating the fold on the string with the degree of freedom corresponding to the particle in the adjoint representation. Along the same idea, 4D QCD has gluons in the adjoint representation that can play the role of the folds, as conjectured a long time ago [14], and used successfully in the phenomenology of gluon jets in the form of 2D folded strings [15].

- The interactions of the folded strings can be inferred from those of the gauge theory[1]. The old and new QCD-string interactions are different from those provided by the Polyakov procedure in standard string theory. Furthermore, there has been a new attempt at the quantization of folded strings in the path integral formalism [16]. There a prescription is given for including folds in a generalization of the Polyakov path integral that describes interacting strings, and a correspondence to QCD is explored. This attempt seems to be related to the approach suggested in [2], since the new modifications of the measure give rise to an effective action that includes the worldline action for the folds in addition to the string action. The path integral approach is of interest

[1]This can be done by examining the 'tHooft-like QCD equations derived in the theory of gluons interacting with adjoint fermions. There are two parts in these equations [14] [12], a zeroth order part that defines the spectrum, and an interaction part. These can be associated with operators in the canonical formulation of folds in the lightcone gauge. The first part describes bound states of n adjoint fermions, and it has a form that is identical to the *quantum eigenvalue equations* for the spectrum of the operator $P_0^- = \sum m_i^2/2p_i^+ + \gamma|x_i^- - x_{i+1}^-|$ that describes n folds in string theory in the lightcone gauge, as derived in [2][3][6]. The second part, called P_{int}^-, describes interactions among the wavefunctions with different numbers of adjoint fermions that represent the folds. The string version of this interaction has not been made precise yet, but is expected to be similar to the one involving end points [7].

mainly because it is promising for the formulation of interacting folded strings. There are many ways of generalizing the string path integral measure, as well as the action, and more needs to be done.

These comments summarize our feeling of the past 19 years that, in addition to being of interest in mathematical physics and generalizations of string physics, folded strings are important in the further development of the string-QCD connection.

In this paper we report on further developments in classical 2D string theory along the direction of [4]. The classical solutions are useful for interpreting the theory of folded strings in curved space-time and also helpful for formulating and understanding the quantum theory. In sections 2,3 we first give new systematic results for the classical folded string solutions by deriving the general form of the "transfer operation" for any 2D target space curved metric, and in sections 4,5,6 we apply the general formalism explicitly to black hole and cosmological metrics. In the literature on classical solutions of strings in curved space-time there exists solutions in higher dimensions which are of a different nature [18] than the ones discussed here.

2   String solutions in 2D curved space-time

Consider a string $x^\mu(\tau, \sigma)$ propagating in a 2D curved space-time manifold characterized by the target space metric $G_{\mu\nu}(x)$. In the conformal gauge, the classical action is given by $\int d^2\sigma \, G_{\mu\nu}(x)\partial_+ x^\mu \partial_- x^\nu$. In the classical theory one can ignore the dilaton (since the dilaton becomes important for conformal invariance only in the quantum theory at higher orders of $\hbar$) . In 2D target space-time the antisymmetric tensor $B_{\mu\nu}(x)$ can be eliminated since it produces a total derivative in the action, and the most general metric can always be transformed into the conformal form $G_{\mu\nu} = \eta_{\mu\nu}G(x)$. Then the most general 2D *classical* string equations of motion and conformal (Virasoro) constraints can be put into the form

$$
\begin{aligned}
\partial_+(G\ \partial_- u) + \partial_-(G\ \partial_+ u) &= \tfrac{\partial G}{\partial v}(\partial_+ u \partial_- v + \partial_+ v \partial_- u) \\
\partial_+(G\ \partial_- v) + \partial_-(G\ \partial_+ v) &= \tfrac{\partial G}{\partial u}(\partial_+ u \partial_- v + \partial_+ v \partial_- u) \\
\partial_+ u \partial_+ v = 0 &= \partial_- u \partial_- v \ ,
\end{aligned}
\tag{1}
$$

where we have used the target space lightcone coordinates

$$u(\sigma^+, \sigma^-) = \frac{1}{\sqrt{2}}(x^0 + x^1), \quad v(\sigma^+, \sigma^-) = \frac{1}{\sqrt{2}}(x^0 - x^1), \qquad (2)$$

and the world sheet lightcone coordinates

$$\sigma^\pm = (\tau \pm \sigma)/\sqrt{2}, \qquad \partial_\pm = (\partial_\tau \pm \partial_\sigma)/\sqrt{2}. \qquad (3)$$

There is a remaining local conformal invariance of left movers and right movers under $\sigma^\pm \to \sigma'^\pm(\sigma^\pm)$ that allows a gauge choice. These equations may be called the "string geodesic equations" since they are indeed the generalization of the particle geodesic equations

$$\partial_+ \partial_- x^\mu + \Gamma^\mu_{\nu\lambda} \partial_+ x^\nu \partial_- x^\lambda = 0. \qquad (4)$$

Since a typical string state is massive, one should expect that the string will follow *on the average* the trajectory of a *massive* particle.Therefore, to understand the average behavior of the string geodesic it is useful to first consider the solution for the geodesic of a massive particle. The particle geodesic equations follow from the above ones by dimensional reduction. That is, by dropping the σ dependence, $\partial_\pm \to \partial_\tau$, these equations reduce to the point particle geodesic equations. For particles, the last line in (1) imposes the condition for a null geodesic, which is too restrictive for our purpose. If this condition is modified to

$$G\dot{u}\dot{v} = \frac{m^2}{2} \qquad (5)$$

then (1) become the equations for a timelike geodesic for a massive particle with mass m. The zero mass limit may also be considered at the end.

We will provide the explicit solutions to the particle as well as the string equations. As discovered in [4] there are additional *stringy phenomena due to the wave nature* that cannot be seen in the particle solution, and therefore it is useful to contrast the string solutions with the particle solutions.

In seeking classical solutions to the string equations one must impose also the properties of periodicity and forward propagation that are required on physical grounds [4]:

(i) The solution must be periodic in the variable σ, $x^\mu(\tau, \sigma) = x^\mu(\tau, \sigma + 4)$,

(ii) Despite the periodicity in σ, the global time coordinate $T(\tau, \sigma) = (u + v)/\sqrt{2}$ must always increase as a function of the proper time τ, for all values of σ.

In flat space-time the solutions take the form

$$x^\mu(\tau, \sigma) = x^\mu_L(\sigma^+) + x^\mu_R(\sigma^-).$$

One can fix the remaining conformal invariance $\sigma^\pm \rightarrow \sigma'^\pm(\sigma^\pm)$ by choosing the gauge, $x^0_L(\sigma^+) = q^0_L + p^0\sigma^+$, $x^0_R(\sigma^-) = q^0_R + p^0\sigma^-$, such that $x^0 \equiv T(\tau, \sigma) = T_0 + p^0\tau$. This form satisfies automatically requirement (ii). In curved space-time, in the conformal gauge, it is not always possible to make $T(\tau, \sigma)$ only a function of τ, but still, on physical grounds, only the solutions in which it does not decrease as a function of τ (for any σ) can be admitted. The two properties (i) and (ii) must be maintained for any physical solution in curved space-time as explained in [4].

In flat space-time $(G = 1)$ the generalized BBHP solution for strings is

$$
\begin{aligned}
u(\sigma^+, \sigma^-) &= u_0 + \tfrac{p^+}{2}\left[(\sigma^+ + f(\sigma^+)) + (\sigma^- - g(\sigma^-))\right] \\
v(\sigma^+, \sigma^-) &= v_0 + \tfrac{p^-}{2}\left[(\sigma^+ - f(\sigma^+)) + (\sigma^- + g(\sigma^-))\right]
\end{aligned}
\tag{6}
$$

where $f(\sigma^+)$ and $g(\sigma^-)$ are any two *periodic functions,* $f(\sigma^+) = f(\sigma^+ + \sqrt{2})$, $g(\sigma^-) = g(\sigma^- + \sqrt{2})$, with slopes $f'(\sigma^+) = \pm 1$ and $g'(\sigma^-) = \pm 1$. The slope can change discontinuously any number of times at arbitrary locations σ^+_i, σ^-_j within the basic intervals $-1/\sqrt{2} \le \sigma^\pm \le 1/\sqrt{2}$ (and then repeated periodically), but the functions f, g are continuous at these points. The discontinuities in the slopes are allowed since the equations of motion are first order in either ∂_+ or ∂_-. The number of times the slope changes in the basic interval corresponds to the number of folds for left movers and right movers respectively. This is seen by taking a snapshot of the string at a constant value of T (i.e. $\tau = \tau_0 = const.$, in the present case of flat space-time), which is easily done by plotting the space coordinate of the string $X(\tau_0, \sigma) = (u - v)/\sqrt{2}$ as a function of σ. The plot shows that the string is folded precisely at the points $\sigma_i(\tau_0)$ where the functions f, g change slopes.

The simplest BBHP solution in flat space-time is the so called yo-yo solution

$$
\begin{aligned}
u(\sigma^+, \sigma^-) &= u_0 + \tfrac{p^+}{2}\left[(\sigma^+ + |\sigma^+|_{per}) + (\sigma^- - |\sigma^-|_{per})\right] \\
v(\sigma^+, \sigma^-) &= v_0 + \tfrac{p^-}{2}\left[(\sigma^+ - |\sigma^+|_{per}) + (\sigma^- + |\sigma^-|_{per})\right]
\end{aligned}
\tag{7}
$$

where $f(\sigma^+) = |\sigma^+|_{per}$, $g(\sigma^-) = -|\sigma^-|_{per}$ are the absolute values of $\sigma^{\pm}$ taken in the basic intervals $-1/\sqrt{2} \leq \sigma^{\pm} \leq 1/\sqrt{2}$, and then repeated periodically. If σ is taken in the full interval $-2 \leq \sigma \leq 2$, then this solution describes the motion of a closed string with two folds, or if σ is taken in the interval $0 \leq \sigma \leq 2$ then it describes an open string without folds but with two end points. The minimal surface swept by the string for the yo-yo solution is plotted in Fig.1. For an open string the minimal surface has a single sheet. For a closed string it consists of two sheets on top of each other, hence folded at the edges.

As a second example consider the same form as eq.(7) but with different periods for $|\sigma^+|_{per+}$ and $|\sigma^-|_{per-}$. For example, take $per^- = \frac{1}{n}per^+$. Then there are $n+1$ critical points that move at the speed of light. In Fig.2 the case of $n = 2$, with three critical points is depicted. Two of these points (the points at the ends at any τ) are folds, but the third point is a saddle point (in the plot of $x(\sigma, \tau_0)$ at fixed τ_0) where the string attempts to fold. Evidently more folds or critical points are generated by more complicated choices of $f(\sigma^+)$ and $g(\sigma^-)$. For additional plots of more complicated solutions, including a discussion of relations among different ways of deriving such solutions, the reader should consult the papers by BBHP.

As suggested in [4], except for deformations due to curvature and singularities, the minimal surfaces in curved space-time are analogous, and they reduce precisely to the BBHP ones in the asymptotically flat regions of space-time where $G \rightarrow 1$.

In curved space-time the general solution of eqs.(1) fall into four classes A, B, C, D

$$
\begin{array}{llll}
A: & u = U(\sigma^+), & v = \bar{V}(\sigma^-) & \\
B: & u = \bar{U}(\sigma^-), & v = V(\sigma^+) & \\
C: & u = u_0, & v = W[\alpha(\sigma^+) + \bar{\beta}(\sigma^-), \ u_0] & (8) \\
D: & u = \bar{W}[\bar{\alpha}(\sigma^-) + \beta(\sigma^+), \ v_0], & v = v_0 \ , &
\end{array}
$$

where $U(\sigma^+), V(\sigma^-), \bar{U}(\sigma^-), \bar{V}(\sigma^+), \alpha(\sigma^+), \beta(\sigma^+), \bar{\alpha}(\sigma^-), \bar{\beta}(\sigma^-)$ are arbitrary and u_0, v_0 are constants[2]. Solutions A, B are present for any metric, but

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[2]This set of solutions were noticed independently in [19] and [20] [11], but the authors of [19] did not realize that the validity of these solutions is limited to patches of the worldsheet, and they assumed that the stringy solutions discussed here and in [4] are gauged away by using the remaining conformal invariance.

the functions $W, \bar{W}$ in solutions C, D are obtained by inverting the following relations that depend on the metric G

$$C: \quad u = u_0, \qquad F(u_0, W) \equiv \int^W dv' G(u_0, v') = \alpha(\sigma^+) + \bar{\beta}(\sigma^-) ,$$

$$D: \quad v = v_0, \qquad \bar{F}(\bar{W}, v_0) \equiv \int^{\bar{W}} du' G(u', v_0) = \bar{\alpha}(\sigma^-) + \beta(\sigma^+),$$

(9)

where the integration is performed at constant $u = u_0$ for solution C, and at constant $v = v_0$ for solution D. Taking derivatives $\partial_\pm$ of the integrals in eq.(9) gives relations that solve the equations (1). So, for a given metric $G(u, v)$ there exists the functions $F(u, v)$ and $\bar{F}(u, v)$ such that their partial derivatives reproduce the metric

$$\frac{\partial F(u, v)}{\partial v} = G(u, v) = \frac{\partial \bar{F}(u, v)}{\partial u}.$$

(10)

and for each metric G we have the relations

$$\begin{aligned}
F(u_0, v) = \alpha + \bar{\beta} &\quad \leftrightarrow \quad v = W(\alpha + \bar{\beta}, u_0), \\
\bar{F}(u, v_0) = \bar{\alpha} + \beta &\quad \leftrightarrow \quad u = \bar{W}(\bar{\alpha} + \beta, v_0),
\end{aligned}$$

(11)

that help define the solutions C, D in terms of the arbitrary functions $\alpha(\sigma^+)$, $\beta(\sigma^+)$, $\bar{\alpha}(\sigma^-)$, $\bar{\beta}(\sigma^-)$. Consider the following three cases as illustrations

1. Flat metric $ds^2 = du\, dv$:

$$\begin{aligned}
F = u_0 + v = \alpha + \bar{\beta}, &\quad \leftrightarrow \quad v = W = \alpha + \bar{\beta} - u_0, \\
\bar{F} = u + v_0 = \bar{\alpha} + \beta, &\quad \leftrightarrow \quad u = \bar{W} = \bar{\alpha} + \beta - v_0.
\end{aligned}$$

(12)

2. SL(2,R)/R black hole metric $ds^2 = (1 - uv)^{-1} du\, dv$:

$$\begin{aligned}
F = -u_0^{-1} \ln(1 - u_0 v) = \alpha + \bar{\beta} \\
\leftrightarrow \quad v = W = u_0^{-1}\{1 - \exp[-u_0(\alpha + \bar{\beta})]\}, \\
\bar{F} = -v_0^{-1} \ln(1 - u v_0) = \bar{\alpha} + \beta \\
\leftrightarrow \quad u = \bar{W} = v_0^{-1}\{1 - \exp[-v_0(\bar{\alpha} + \beta)]\}.
\end{aligned}$$

(13)

3. Cosmological (de Sitter) metric $ds^2 = dt^2 - e^{2Ht} dx^2 = \frac{4}{H^2}(u+v)^{-2} du\, dv$:

$$\begin{aligned}
F = -(u_0 + v)^{-1} = \alpha + \bar{\beta} &\quad \leftrightarrow \quad v = W = -(\alpha + \bar{\beta})^{-1} - u_0, \\
\bar{F} = -(u + v_0)^{-1} = \bar{\alpha} + \beta &\quad \leftrightarrow \quad u = \bar{W} = -(\bar{\alpha} + \beta)^{-1} - v_0.
\end{aligned}$$

(14)

10

Since one still needs to impose the periodicity and forward propagation conditions (i) and (ii) given above, (8-14) are not yet legitimate solutions. As discussed in [4] each one of the forms A, B, C, D in (8) can be valid only in certain patches of the world sheet (σ^+, σ^-), and these solutions need to be matched to each other at the boundaries of the patches. Thus, to construct a legitimate solution in curved space-time, first one must decide on the form of the solution in the flat asymptotic region by making a choice for the functions f, g in eq.(6). This is a boundary condition which is consistent with the requirements (i) and (ii). The sign patterns of the derivatives $(f', g') = (+, -), (-, +), (+, +), (-, -)$ divides the world sheet (σ^+, σ^-) into patches where the corresponding signs hold. To each such patch the forms A, B, C, D in (8) are assigned respectively, for any curved metric G. The pattern of assigned forms must be periodic in the direction of σ but not in the direction of τ. The patterns A, B, C, D are the same for flat or curved space-time. The difference between curved and flat space-time arises in the choice of the metric dependent functions $W, \bar{W}$ for the C, D patches. Then these solutions are matched at the boundaries between patches. This procedure insures the properties (i) and (ii) in curved space-time while being consistent with boundary conditions (i.e. some given BBHP solution) in the asymptotically flat region of the target space metric G.

It turns out that the constant values of u or v in the C or D patches respectively provide sufficient data for constructing the motion of the entire string (see below for details). The solutions in these patches describe the *motion of the folds*[3]. Specifically note that a constant value of u or v describes

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[3]By definition, at a fold the determinant of the induced metric, $g_{\alpha\beta} = \partial_\alpha x^\mu \partial_\beta x^\nu G_{\mu\nu}$, vanishes, $\det g = 0$. But in the conformal gauge the induced metric itself also vanishes locally everywhere since $g_{\alpha\beta} = \Lambda \eta_{\alpha\beta}$. Note that this does not mean that the world sheet metric $\eta_{\alpha\beta}$ vanishes. In the C, D cells by virtue of having either u or v constant throughout the cell one gets $g_{\alpha\beta} = 0$, indicating that all points in these cells are mapped to the trajectory of the fold in target spacetime. The mapping is many to one, since a region of the world sheet is mapped to a segment (trajectory of the fold) in target spacetime. Therefore, a fold in target spacetime has many representatives on the world sheet. For example, consider the leftmost C-type cell at the bottom of the diagram in (16) for which $u = u_{k-1}$, and $v = W_{k-1}(\sigma^+, \sigma^-)$. At a constant $\tau = \tau_0$, all the σ points that give the same value of $v = v_0$ are mapped to the same fold located at (u_{k-1}, v_0). As τ changes $u = u_{k-1}$ remains fixed while v changes along the lightlike trajectory of the fold. To trace the trajectory of a fold it is sufficient to concentrate on one of its images on the world sheet. Such representative images are the vertical lines at $\sigma = 0, 2$ in the diagram in (16).

a light-like trajectory, indicating that a fold moves at the speed of light. In order to determine these constants in different time intervals, one must go through the procedure of matching boundaries at the patches as defined in the previous paragraph.

3 The yo-yo in any curved space-time

As an illustration of the procedure we consider the simplest boundary condition in the asymptotic region, namely the yo-yo solution given in (7). The pattern that emerges is as follows. The world sheet is labelled by σ horizontally and by τ vertically. It is sliced by equally spaced 45° lines that form a light-cone lattice in $\sigma^\pm$. The crosses in the diagram represent the corners of the cells on the world sheet. Each cell on the world sheet is labelled by the values of (σ^+, σ^-) at the center of the cell, divided by a factor of $\sqrt{2}$. For example at the center of the cell labelled by (m, n) the world sheet coordinates are $\sigma^+ = m\sqrt{2}, \quad \sigma^- = n\sqrt{2}$, and the (τ, σ) coordinates are $\tau = m + n, \quad \sigma = m - n$. The points inside the cell (m, n) are parametrized by $\sigma^\pm$ in the ranges

$$(m - \frac{1}{2})\sqrt{2} < \sigma^+ < (m + \frac{1}{2})\sqrt{2}, \qquad (n - \frac{1}{2})\sqrt{2} < \sigma^- < (n + \frac{1}{2})\sqrt{2}. \quad (15)$$

The A, B, C, D solutions that are placed into these cells provide a map from the world sheet to the target space-time.

$\sigma = 0$	$\sigma = 1$	$\sigma = 2$	$\sigma = 3$	$\sigma = 4 \equiv 0$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\times$	$U_{k+2}(\sigma^+)$ $V_{k+2}(\sigma^-)$	$\times$	$U_{k+2}(\sigma^-)$ $V_{k+2}(\sigma^+)$	$\times$
$\cdots$ $\begin{matrix} u_{k+1} \\ W_{k+1}(\sigma^+, \sigma^-) \end{matrix}$	$\times$	$\begin{matrix} \bar{W}_{k+1}(\sigma^+, \sigma^-) \\ v_{k+1} \end{matrix}$	$\times$	$\begin{matrix} u_{k+1} \\ W_{k+1}(\sigma^+, \sigma^-) \end{matrix}$ $\cdots$
$\times$	$U_{k+1}(\sigma^-)$ $V_{k+1}(\sigma^+)$	$\times$	$U_{k+1}(\sigma^+)$ $V_{k+1}(\sigma^-)$	$\times$
$\cdots$ $\begin{matrix} \bar{W}_k(\sigma^+, \sigma^-) \\ v_k \end{matrix}$	$\times$	$\begin{matrix} u_k \\ W_k(\sigma^+, \sigma^-) \end{matrix}$	$\times$	$\begin{matrix} \bar{W}_k(\sigma^+, \sigma^-) \\ v_k \end{matrix}$ $\cdots$
$\times$	$U_k(\sigma^+)$ $V_k(\sigma^-)$	$\times$	$U_k(\sigma^-)$ $V_k(\sigma^+)$	$\times$
$\cdots$ $\begin{matrix} u_{k-1} \\ W_{k-1}(\sigma^+, \sigma^-) \end{matrix}$	$\times$	$\begin{matrix} \bar{W}_{k-1}(\sigma^+, \sigma^-) \\ v_{k-1} \end{matrix}$	$\times$	$\begin{matrix} u_{k-1} \\ W_{k-1}(\sigma^+, \sigma^-) \end{matrix}$ $\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

$$(16)$$

Consider all the cells in a horizontal row corresponding to a fixed value of τ. There are two types of cells, the A, B type whose centers are at $\tau = m + n = 2k$ (i.e. $m = k + l$, $n = k - l$), and the C, D type cells whose centers are at $\tau = m + n = 2k + 1$ (i.e. $m = k + l + 1$, $n = k - l$ or $m = k + l$, $n = k - l + 1$). Corresponding to the A, B, C, D patterns, the solutions in (8) are assigned periodically in σ as follows:

- All the A cells whose centers are at $\tau = 2k$, $m = even$, $n = even$ are assigned periodically the same solution $u = U_k(\sigma^+)$, $v = V_k(\sigma^-)$,

- All the B cells whose centers are at $\tau = 2k$, $m = odd$, $n = odd$ are assigned periodically the same solution[4] $u = U_k(\sigma^-)$, $v = V_k(\sigma^+)$,

- All the C cells whose centers are at $\tau = 2k + 1$, $m = odd$, $n = even$ are assigned periodically the same solution $u = u_k$, $v = W_k(\sigma^+, \sigma^-)$,

- All the D cells whose centers are at $\tau = 2k + 1$, $m = even$, $n = odd$ are assigned periodically the same solution $u = \bar{W}_k(\sigma^+, \sigma^-)$, $v = v_k$

By assigning the *same function* to all the cells of the same type at a fixed τ (or fixed k) one obtains a pattern that insures periodicity under $\sigma \to \sigma + 4$, (or $l \to l + 2$). This periodicity may also be insured by taking periodic functions $U_k(z + \sqrt{2}) = U_k(z)$ and $V_k(z + \sqrt{2}) = V_k(z)$. For different k (i.e. different τ) the functions U_k, V_k, etc. are different, but are related to each other by matching boundary conditions across the cell boundaries. Therefore, this procedure corresponds to a world-sheet with the topology of a cylinder. The map provided by the functions is from the cylinder to curved space-time whose metric is $G_{\mu\nu} = \eta_{\mu\nu} G(u, v)$.

The continuity at the corners that join the A, B cells is automatically insured by the use of the same functions $U_k(z), V_k(z)$ to describe the A, B solutions (but with different arguments $z = \sigma^\pm$ that alternate between neighboring cells, see footnote). Continuity at the boundaries between A, B cells

---

[4] A priori the U, V solutions are represented by different functions in the A, B patches at the same τ. The convenient use of the same set of functions $U_k(z), V_k(z)$ for both the A, B type cells (but with $z = \sigma^\pm \to \sigma^\mp$) corresponds to fixing the remaining conformal gauge partially. One is allowed to choose a gauge locally as long as this is not in conflict with the matching of boundary conditions.

and C, D cells requires

$$U_{k+1}(-1/\sqrt{2}) = U_k(1/\sqrt{2}) = u_k,$$
$$V_{k+1}(-1/\sqrt{2}) = V_k(1/\sqrt{2}) = v_k. \tag{17}$$

where the (u_k, v_k) are constants. Similarly, by taking into account the relations (11) at these boundaries one can construct the functions $W_k, \bar{W}_k$ for the C, D cells in terms of the functions U_k, V_k

$$W_k(\sigma^+, \sigma^-) = W\left[(F(u_k, V_k(\sigma^+)) + F(u_k, V_k(\sigma^-)) - F(u_k, v_{k-1})), u_k\right]$$
$$\bar{W}_k(\sigma^+, \sigma^-) = \bar{W}\left[\left(\bar{F}(U_k(\sigma^+), v_k) + \bar{F}(U_k(\sigma^-), v_k) - \bar{F}(u_{k-1}, v_k)\right), v_k\right]. \tag{18}$$

Evaluating these at the lower (i.e. past) boundaries of the C, D cells, using $V_k(-1/\sqrt{2}) = v_{k-1}$, $U_k(-1/\sqrt{2}) = u_{k-1}$, the boundary matching is insured by the fact that F and W are inverses of each other (see eq.(11-14)

$$W(F(u_k, V_k(z)), u_k) = V_k(z), \tag{19}$$

and similarly for $U_k(z)$. At the upper (i.e. future) boundaries of the C, D cells the boundary matching gives a *recursion relation*

$$V_{k+1}(z) = W\left[(F(u_k, V_k(z)) + F(u_k, v_k) - F(u_k, v_{k-1})), u_k\right]$$
$$U_{k+1}(z) = \bar{W}\left[\left(\bar{F}(U_k(z), v_k) + \bar{F}(u_k, v_k) - \bar{F}(u_{k-1}, v_k)\right), v_k\right], \tag{20}$$

where $z = \sigma^\pm$. This recursion may be viewed as a *transfer* operation in proper time $\tau \to \tau + 2$, for any σ, and is quite analogous to the concept of the "transfer matrix" in lattice theories. The recursion leads to the solution of all the $U_k(\sigma^\pm)$, $V_k(\sigma^\pm)$ in terms of $U_0(z)$, $V_0(z)$, that describe initial conditions at $\tau = 0$.

By evaluating the recursion relation (20) at the boundaries of each cell $z = \pm 1/\sqrt{2}$ and using the values (17) at the boundaries, one finds a recursion relation for the constants (u_k, v_k)

$$v_{k+1} = W\left[(2F(u_k, v_k) - F(u_k, v_{k-1})), u_k\right],$$
$$u_{k+1} = \bar{W}\left[\left(2\bar{F}(u_k, v_k) - \bar{F}(u_{k-1}, v_k)\right), v_k\right]. \tag{21}$$

The solution of this recursion relation requires 4 initial constants u_0, v_0, u_{-1}, v_{-1}

$$U_0(-1/\sqrt{2}) = u_{-1} \quad U_0(1/\sqrt{2}) = u_0 \quad,$$
$$V_0(-1/\sqrt{2}) = v_{-1} \quad V_0(1/\sqrt{2}) = v_0 \quad. \tag{22}$$

14

Therefore, the positions (u_k, v_k) are fully determined in curved space-time in terms of 4 initial constants.

The constants (u_k, v_k) are sufficient to describe the physical motion of the folds (or end points), as well as the whole string, as follows. Consider the diagram of eq.(16). At any τ the trajectories of the folds are parametrized by the vertical lines that pass through $\sigma = 0, 2$ on the world sheet (and their periodic repetitions at $\sigma = 4l$, $4l + 2$, see footnote). Likewise, vertical lines that pass through the crosses located at $\sigma = 1, 3$ (and their periodic repetitions at $\sigma = 4l + 1$, $4l + 3$) parametrize the trajectory of the midpoint between the folds. The center of mass of the string coincides with these midpoints. As τ increases one can read off the space-time trajectories of the center of mass and of the folds by moving upward along the vertical lines in the diagram. For example, consider the $\sigma = 0$ fold: during $2k - 2 \leq \tau \leq 2k$ it remains at constant $u = u_{k-1}$ while the value of $v = W_{k-1}$ increases from $v = v_{k-2}$ to $v = v_k$. Between $2k \leq \tau \leq 2k + 2$ it remains at constant $v = v_k$ while the value of $u = \bar{W}_k$ increases from $u = u_{k-1}$ to $u = u_{k+1}$, etc. In a similar way the trajectory of the second fold and of the center of mass are read off directly from the diagram in eq.(16). The space-time trajectories of these points are plotted in a (u, v) plot in Fig.3.

The detailed motion of the intermediate points of the string at any σ are described by the functions $U_k, V_k, W_k, \bar{W}_k$ as indicated on the diagram (16) and mapped on Fig.3. The space-time trajectories of folds or end points that are the images of $\sigma = 0, 2$ are physical and cannot depend on conformal reparametrizations. Indeed, as seen from the above solution there is no freedom in the choice of the constants (u_k, v_k) except for the initial values (22). On the other hand, the motion of the rest of the string is gauge dependent at intermediate points σ (because of reparametrizations), and therefore it depends on the choice of $U_0(z), V_0(z)$ that have remained unspecified. However, once the motion of the end points is plotted, it is clear from Fig.3 that the shape of the minimal surface is already determined without needing the details of the gauge dependent motion of the intermediate points.

The remaining conformal invariance may be used to fix the form of these functions in the initial cell (although this is not necessary). For the yo-yo solution the initial functions $U_0(z)$, $V_0(z)$ need not contain more than 4 constants that are related to the initial positions and velocities of the two

15

folds. Therefore, the simplest gauge fixed form is

$$U_0(z) = \tfrac{1}{2}(u_0 + u_{-1}) + \tfrac{1}{\sqrt{2}}(u_0 - u_{-1})\, z_{per}$$
$$V_0(z) = \tfrac{1}{2}(v_0 + v_{-1}) + \tfrac{1}{\sqrt{2}}(v_0 - v_{-1})\, z_{per}, \tag{23}$$

where z_{per} is the linear function $z_{per} = z$ in the interval $-1/\sqrt{2} \leq z \leq 1/\sqrt{2}$, and then repeated periodically. However, any other periodic function with the same 4 boundary constants will produce the same physical motion for the folds.

The recursion (21) is the fundamental physical relation that fully determines the motion of the yo-yo string in curved space-time. We called it the "transfer matrix" in the example of the black hole worked out in ref.[4]. It was found that it has certain invariances that are valid everywhere in target space-time, including near singularities. The invariance is related to a lattice version of the fundamental action $A = \int d^2\sigma\, G_{\mu\nu} \partial_+ x^\mu \partial_- x^\nu$ that represents the minimal surface swept by the string. The lattice version of the minimal surface is expressed in terms of the constants (u_k, v_k), and its value for one period turns out to be a constant of motion. Explicit expressions for this lattice action will be given for specific metrics in the following sections. For every metric G one can find a lattice version of the action A that is an invariant under the recursion (21). The invariance is valid even in the vicinity of singularities in space-time (i.e. when $G(u_k, v_k)$ grows) and helps in the understanding of new stringy phenomena. For example, it was found that classical strings can tunnel to regions of space-time (such as the bare singularity region of a black hole) that are forbidden to particle geodesics. Such a surprising motion of a string may be thought of as the analog of the diffraction of light around corners, that is possible for classical waves, but is impossible for particle trajectories.

In this section we constructed the yo-yo solution in any curved space-time given by G. In a similar way one may consider more complicated solutions with many folds. The general boundary condition near $G \to 1$ given by (6), with any number folds, defines a pattern of A, B, C, D on the world sheet that corresponds to the regions of (σ^+, σ^-) that have definite signs of f', g' for some choice of f, g. The pattern must be periodic horizontally, with a period of $\sigma \to \sigma + 4$, to insure periodicity. This generalizes the lattice in the diagram of (16). By virtue of the BBHP construction, any of these generalized patterns is guaranteed to correspond to strings that propagate forward

16

in time. Then there remains to carry out the matching of the functions at the boundaries. This would give generalizations of the recursion relations and transfer matrices discussed above. It seems that this is a very rich area for mathematical physics, since one may explore relations between geometries defined by metrics G, lattices, and transfer matrices. It is clear that the general behavior of the minimal surface that emerges from this procedure has to be quite similar to the one in flat space-time (which is already given by the choice of f, g), except for the deformations due to curvature and singularities. Moreover, it seems that the main physical stringy features related to the curvature and/or singularity structure of space-time may already be extracted from the yo-yo solution that has only two folds.

We now apply the general yo-yo results to several specific metrics and construct explicitly the corresponding "transfer matrices", their invariants, and the corresponding string solutions.

4   Flat Space-time

The functions $\bar{F}, \bar{W}$ corresponding to the flat space-time metric $G = 1$ are given in (12). Using them in the general formulas (17-22) we obtain the explicit recursion relations

$$
\begin{aligned}
\bar{W}_k &= U_k(\sigma^+) + U_k(\sigma^-) - u_{k-1}, \\
U_{k+1}(z) &= U_k(z) + u_k - u_{k-1} \\
u_{k+1} &= 2u_k - u_{k-1}
\end{aligned}
\tag{24}
$$

They are solved by

$$
\begin{aligned}
u_k &= u_0 + k(u_0 - u_{-1}) \\
U_k(z) &= U_0(z) + k(u_0 - u_{-1}) \\
\bar{W}_k &= U_0(\sigma^+) + U_0(\sigma^-) + (k+1)(u_0 - u_{-1}) - u_0
\end{aligned}
\tag{25}
$$

where $U_0(-1/\sqrt{2}) = u_{-1}$, $U_0(1/\sqrt{2}) = u_0$, and the function $U_0(z)$ is arbitrary. The solutions for $V_k(z), W_k, v_k$ are obtained from the above by replacing $U \to V$ and $u \to v$. If $U_0(z), V_0(z)$ are gauge fixed as in (23), then this solution takes the convenient form of the BBHP yo-yo string in (7). The present form is a generalization that permits other gauge choices. The motion of the end points, as plotted in Fig.1 is gauge independent, but the

17

motion of the interior points of the string depends on the gauge choice, as expected.

Define a lattice version of the surface element $dA = d^2\sigma\,(\partial_+ u\,\partial_- v + \partial_- u\,\partial_+ v)$ swept by the string during $2k \leq \tau \leq 2k + 2$. The area of one rectangle in Fig.1 is

$$dA_k = (u_k - u_{k-1})\,(v_k - v_{k-1}). \tag{26}$$

From the world sheet point of view this covers the image of one A or B cell, while the image of a C or D cell has zero area in target space-time (since they are mapped to the edges of the rectangle). Consider the transformation (24) as a transfer matrix that takes the system forward in time. Under this transformation dA_k is an invariant since $dA_{k+1} = dA_k$. This is seen by rewriting (24) in the form $U_{k+1}(z) - u_k = U_k(z) - u_{k-1}$, etc.. Therefore, we may say that the "transfer matrix" for flat space-time given by (24) leaves invariant the "lattice action density" given by (26). This concept generalizes to curved space-time, as seen below.

5   Black hole space-time

The case of the SL(2,R)/R two dimensional black hole metric $ds^2 = (1 - uv)^{-1}du\,dv$ was already discussed in [4], but here we will show how the results of [4] follow from the general formulas, and also give the additional recursion relations for $U_k, V_k, W_k, \bar{W}_k$ at general k and general gauge that were not provided in [4].

The solution for the geodesic of a massive particle was given in our previous work [20] [4]. Here we rewrite it in a more convenient form

$$
\begin{aligned}
u(\tau) &= e^{\sqrt{\gamma^2 + m^2/2}\,\tau}\left[u_0\cosh(\gamma\tau) - \left(u_0\sqrt{\gamma^2 + m^2/2} - \dot{u}_0\right)\tfrac{1}{\gamma}\sinh(\gamma\tau)\right] \\
v(\tau) &= e^{-\sqrt{\gamma^2 + m^2/2}\,\tau}\left[v_0\cosh(\gamma\tau) + \left(v_0\sqrt{\gamma^2 + m^2/2} + \dot{v}_0\right)\tfrac{1}{\gamma}\sinh(\gamma\tau)\right],
\end{aligned}
\tag{27}
$$

where $u_0, v_0, \dot{u}_0, \dot{v}_0$ are initial velocities and momenta, m is the mass of the particle, and γ is a convenient parameter

$$\gamma = \frac{\sqrt{(u_0\dot{v}_0 + \dot{u}_0 v_0)^2 - 4\dot{u}_0\dot{v}_0}}{2(1 - u_0 v_0)}, \qquad \frac{\dot{u}_0\dot{v}_0}{(1 - u_0 v_0)} = \frac{m^2}{2}. \tag{28}$$

In the zero mass limit either $\dot{u}_0 = 0$ or $\dot{v}_0 = 0$, and then the solution reduces to a light-like geodesic for which either u or v remain constant respectively at all times.

The singularity is at $u(\tau)v(\tau) - 1 = 0$. To see when the particle hits the singularity we compute this quantity

$$\frac{u(\tau)v(\tau) - 1}{u_0 v_0 - 1} = \left[\cosh \gamma\tau + \frac{u_0 \dot{v}_0 + \dot{u}_0 v_0}{\sqrt{(u_0 \dot{v}_0 + \dot{u}_0 v_0)^2 - 4\dot{u}_0 \dot{v}_0}} \sinh \gamma\tau \right]^2. \qquad (29)$$

In the massless limit this expression becomes

$$\frac{u(\tau)v(\tau) - 1}{u_0 v_0 - 1} = \exp\left((u_0 \dot{v}_0 + \dot{u}_0 v_0)\tau\right), \quad \dot{u}_0 \dot{v}_0 = 0. \qquad (30)$$

It is evident that the sign of $uv - 1$ cannot change as τ changes, therefore the particle must remain in either the black hole region $uv < 1$ (can cross the horizon at $u = 0$ or $v = 0$), or in the bare singularity region $uv > 1$. The boundary $uv = 1$ acts like an impenetrable wall from either side. This last feature is different for the string solution. In contrast to the point particle, the string will tunnel through the wall !! This surprising effect was discovered in [4].

It was evident from the work of [4] that, except for the tunneling type phenomena, the string follows more or less the geodesic of the *massive* particle. Therefore, it is useful to clarify the properties of the geodesics of the point particle, because they depend on the initial particle location as well as its velocity.

- If the particle starts out in the "bare singularity" region , $u_0 v_0 > 1$ (future or past regions), the mass formula in (28) requires $\dot{u}_0 \dot{v}_0 < 0$ and γ is real. Then the motion is governed by hyperbolic functions, and (29) never vanishes. Therefore, a *massive particle, or the string, cannot hit the singularity*. In the massless limit, according to (30), the light-like geodesic will hit the bare singularity only if it starts out with initial conditions that give $u_0 \dot{v}_0 + \dot{u}_0 v_0 < 0$, but in any case it reaches the singularity only at infinite proper time $\tau = \infty$. Therefore, the "bare singularity" region of the SL(2,R)/R black hole is not a singularity that can be reached by physical signals in a finite amount of proper time. In this sense it is *not really a singularity*.

19

- If the particle starts initially in the black hole region $u_0 v_0 < 1$, either inside or outside the horizon, its trajectory has wildly different behavior depending on its velocity. There are two critical ratios of the velocities at which $\gamma = 0$.

(i) If the velocities lie in the range

$$\left(\frac{1 - \sqrt{1 - u_0 v_0}}{u_0}\right)^2 < \frac{\dot{v}_0}{\dot{u}_0} < \left(\frac{1 + \sqrt{1 - u_0 v_0}}{u_0}\right)^2. \qquad (31)$$

then γ is imaginary and (29) vanishes periodically. The massive particle goes through the horizon and hits the future singularity at a *finite* value of τ. There it moves smoothly to a second sheet of the (u, v) space-time, but still with $uv < 1$. It continues its journey toward the second branch of the singularity and hits it, moving on to a third sheet of space-time (or back to the first sheet, according to interpretation). The journey continues endlessly from singularity to singularity, always moving smoothly to another sheet, and always remaining in the region $uv < 1$. This behavior is similar to the behavior of geodesics in the many worlds of the Reissner-Nordtrom black hole[5].

(ii) If the velocities lie in the range

$$\frac{\dot{v}_0}{\dot{u}_0} > \left(\frac{1 + \sqrt{1 - u_0 v_0}}{u_0}\right)^2 \qquad (32)$$

then γ is real, the motion is hyperbolic, and (29) vanishes only once. Therefore, the particle hits the black hole at a finite τ only once, and moves to a second sheet where it remains for the rest of time.

(iii) If the velocities lie in the range

$$\frac{\dot{v}_0}{\dot{u}_0} < \left(\frac{1 - \sqrt{1 - u_0 v_0}}{u_0}\right)^2 \qquad (33)$$

---

[5] In the present case the worlds are pasted to each other just at $uv = 1$ along the singularity. When the metric is modified by quantum corrections [17] a gap develops so that the singularity becomes unreachable while the geodesics move from one world to the next.

then γ is real, the motion is hyperbolic, but (29) never vanishes. Therefore, the particle never hits the black hole.

The string geodesics given below follows, on the average, the behavior of the massive particle geodesics above. But, because of the oscillatory motion we find new phenomena in the vicinity of the black hole. When the string approaches the black hole from the $uv < 1$ region, and hits the singularity, it behaves differently than the particle: it fully penetrates the wall to the $uv > 1$ region, but then it snaps back into the $uv < 1$ region, and then follows more or less the particle trajectory in the second sheet, etc. (see the solution below and the plots in Figs.4,5).

To construct the string solution we use the general formulas of the previous sections. The functions $\bar{F}, \bar{W}$ corresponding to the flat space-time metric $G = (1 - uv)^{-1}$ are given in (12). Using them in the general formulas (17-22) we obtain the explicit recursion relations

$$\bar{W}_k = \frac{1}{v_k} \left[1 - \frac{\left(1 - U_k(\sigma^+)v_k\right)\left(1 - U_k(\sigma^-)v_k\right)}{1 - u_{k-1}v_k} \right]$$
$$U_{k+1}(z) = \frac{1 - u_k v_k}{1 - u_{k-1}v_k} \left[U_k(z) + \frac{u_k - u_{k-1}}{1 - u_k v_k} \right] \tag{34}$$
$$u_{k+1} = \frac{2u_k - u_{k-1} - u_k^2 v_k}{1 - u_{k-1}v_k},$$

and similarly W_k, V_k, v_k are obtained from the above by interchanging $U \leftrightarrow V$ and $u \leftrightarrow v$. This agrees with the results of [4]. Note that for $u, v \to 0$ or ∞ the metric approaches the flat metric. In both of these limits the formulas in (34) approach the flat ones in (24).

By feeding the recursion relations to a computer, the trajectories of the folds are plotted in Fig.4,5. A physical discussion of the string falling into a black hole was given in [4]. The most surprising effect was the tunnelling of the string into the bare singularity region which is not possible for particles (Fig.5). As suggested before, this is analogous to the diffraction of classical light waves that is possible for waves but not for particles.

Just as the flat case, we define a lattice version of the area element in curved space-time. The "lattice area" swept by the string for one of the rectangles in Fig.4,5 is defined as

$$dA_k = \frac{(u_k - u_{k-1})(v_k - v_{k-1})}{1 - \frac{1}{4}(u_k + u_{k-1})(v_k + v_{k-1})}. \tag{35}$$

21

As in the flat case, this is a lattice version of the target space area of the image of a A or B cell on the world sheet, while the area of the image of a C or D cell is zero. This expression is invariant under the "transfer matrix" (34), i.e. $dA_k = dA_{k+1}$. The invariance of this expression everywhere, including in the vicinity of the black hole singularity, is helpful in understanding the reason for the tunnelling to the bare singularity region. Namely, since the string must move in a way that conserves this minimal area, and must have a continuous trajectory, it cannot avoid the tunnelling for generic initial conditions set by an observer (see Fig.5).

6   Cosmological space-time

Consider the cosmological space-time corresponding to a Friedman - Robertson - Walker (FRW) universe in 4D

$$ds^2 = dt^2 - R^2(t) \left(\frac{dr^2}{1 - kr^2} + r^2 d\Omega^2 \right). \tag{36}$$

where $k = -1, 0, 1$ are related to the classification of cosmological space-times as "open, flat, closed" respectively. For a string moving purely along the radial direction $d\theta = d\phi = 0$ one concentrates on the 2D metric

$$ds^2 = dt^2 - R^2(t) \frac{dr^2}{1 - kr^2}. \tag{37}$$

It is convenient to change variables

$$\sqrt{k}\, r = \sin(\sqrt{k}X), \quad T = \int^t \frac{dt'}{R(t')}, \quad \sqrt{k} = i, 0, 1$$
$$u = \frac{1}{\sqrt{2}}(T + X), \quad v = \frac{1}{\sqrt{2}}(T - X), \tag{38}$$

so that the line element takes the conformal form

$$ds^2 = R^2 \left(dT^2 - dX^2 \right) = G(u, v)\, du dv,$$
$$G(u, v) = 2R^2(t(T)). \tag{39}$$

Once written in terms of (u, v) the complete manifold is usually obtained by analytic continuation to all values of these variables. Then one may apply the general formulas of the previous sections to obtain the classical motion of strings.

As an example consider the de Sitter universe for which the expansion factor of the universe is given by

$$|R(t)| = e^{Ht} \tag{40}$$

where $H = \dot{R}/R$ is the Hubble constant, and

$$ds^2 = \frac{4\,du\,dv}{H^2(u+v)^2}. \tag{41}$$

This 2D space can be embedded in 3D as the surface of a hyperboloid described by

$$x_0^2 - x_1^2 - x_2^2 = -H^{-2}$$
$$x_0 = \frac{uv - H^{-2}}{u+v}, \quad x_1 = \frac{uv + H^{-2}}{u+v}, \quad x_2 = \frac{1}{H}\frac{u-v}{u+v} \tag{42}$$

Then the metric in (41) takes the flat form

$$ds^2 = dx_0^2 - dx_1^2 - dx_2^2. \tag{43}$$

First consider the geodesic equations for a massive particle of mass m. They can be solved exactly as a function of proper time τ

$$u(\tau) = c + \frac{\sinh(Hm\tau) - \sinh(Hm\tau_0)}{H\sinh[Hm(\tau+\tau_0)]}$$
$$v(\tau) = -c - \frac{\sinh(Hm\tau) + \sinh(Hm\tau_0)}{H\sinh[Hm(\tau+\tau_0)]} \tag{44}$$
$$R(\tau) = \frac{\sinh[Hm(\tau+\tau_0)]}{\sinh(Hm\tau_0)} = -\frac{2}{H}\frac{1}{u+v}$$

where c, τ_0 are constants, and $R(\tau = 0) = 1$ has been chosen for simplicity. The geodesic for the massive particle is best pictured on the surface of the hyperboloid $x_1^2 + x_2^2 = x_0^2 + H^{-2}$. Inserting the solution in (42) one sees that $x_0(\tau)$ increases monotonically and lies in the range $-\infty < x_0(\tau) < \infty$. The geodesic extends from a point on the infinitely large circle at $x_0 = -\infty$ to a point on the infinitely large circle at $x_0 = \infty$. It is a line that spirals less than or equal to one time on this surface. Define the angle $\tan\theta = x_2/x_1$. If the mass is zero, the maximum spiralling angle $\Delta\theta = \theta(\infty) - \theta(-\infty)$ is exactly 2π, but for the massive particle the angle is less than 2π.

Thus, on the average, we must expect the string center of mass to spiral less than 2π. Of course, the overall string performs the yo-yo oscillations of

23

Fig.3 and sweeps a minimal surface on the hyperboloid, that is similar to the one in flat space-time except for deformations due to curvature.

The explicit solution that describes this motion is obtained by applying our general procedure that yields the transfer matrix

$$
\begin{aligned}
\bar{W}_k(\sigma^+, \sigma^-) &= \left[\frac{1}{U_k(\sigma^+)+v_k} + \frac{1}{U_k(\sigma^-)+v_k} - \frac{1}{u_{k-1}+v_k} \right]^{-1} - v_k \\
U_{k+1}(z) &= \left[\frac{1}{U_k(z)+v_k} + \frac{1}{u_k+v_k} - \frac{1}{u_{k-1}+v_k} \right]^{-1} - v_k \\
u_{k+1} &= \left[\frac{2}{u_k+v_k} - \frac{1}{u_k+v_{k-1}} \right]^{-1} - v_k
\end{aligned}
\tag{45}
$$

Similar formulas hold for W_k, V_k, v_k respectively. We define a discrete version of the minimal area for rectangle k by

$$
dA_k = \frac{4}{H^2} \frac{(u_k - u_{k-1})(v_k - v_{k-1})}{(u_k + v_{k-1})(u_{k-1} + v_k)}.
\tag{46}
$$

The transfer matrix leaves invariant this discrete minimal area, i.e. $dA_{k+1} = dA_k$. This is easily proven by rewriting the transfer matrix in the form

$$
\begin{aligned}
\frac{U_{k+1}(z)-u_k}{[U_{k+1}(z)+v_k](u_k+v_k)} &= \frac{U_k(z)-u_{k-1}}{[U_k(z)+v_k](u_{k-1}+v_k)} \\
\\
\frac{V_{k+1}(z)-v_k}{[u_k+V_{k+1}(z)](u_k+v_k)} &= \frac{V_k(z)-v_{k-1}}{[u_k+V_k(z)](u_k+v_{k-1})}.
\end{aligned}
\tag{47}
$$

By feeding the recursion relation to a computer, and plotting the trajectories of the folds, the minimal surface is constructed and seen to have the properties described above, as depicted in Fig.6

7 Comments and Conclusions

We have solved generally the classical 2D string theory in any curved space-time. All stringy solutions correspond to folded strings. All solutions tend to the BBHP solutions in the asymptotically flat region of the curved space-time. Therefore, the BBHP solutions of eq.(6) serve to classify all the solutions for any curved space-time. In fact, the sign patterns of the BBHP solutions provide the method for dividing the world-sheet into patches, thus defining the lattices associated with the A, B, C, D solutions, as explained in the paragraph following eq.(14). The matching of boundaries for these

functions gives the general solution in curved space-time in the form of a "transfer matrix". Thus, lattices on the world-sheet plus geometry in space-time lead to transfer matrices. This seems to be a rich area to explore in more detail.

The method was explicitly applied to the yo-yo solution, and the general yo-yo solution in any curved space-time was constructed. Specializing further the metric, the transfer matrices were derived for a black hole space-time and for a cosmological space-time.

The general physical motion of the string follows on the average a geodesic of the massive particle, consistent with intuition. However, the stringy behavior becomes evident in the vicinity of singularities where new phenomena, such as tunneling (similar to diffraction), take place.

Given the fact that the string in 2D is quite non-trivial classically, we expect that there is a consistent quantization procedure that includes the non-trivial folded states. We have already outlined in the introduction that in covariant quantization (as well as in the semiclassical quantization of folds carried out by BBHP) the 2D string in flat space-time has indeed extra states corresponding to folded strings. A similar covariant quantization can be carried out for the 2D black hole string by using the Kac-Moody current algebra formulation, and relaxing the $c = 26$ condition (i.e. $k < 9/4$) to include the folded strings. What would also be interesting is to find the correct formulation for interacting folded strings. The path integral approach started in [16] seems to be promising, and it may be possible to make faster progress by reformulating it in the conformal gauge and relating it to our classical solutions[6].

Folded strings exist in higher dimensions as well. One can display the general solution in flat space-time in the temporal gauge

$$x^0 = p^0\tau, \quad \vec{x}(\tau,\sigma) = \vec{x}_L(\sigma^+) + \vec{x}_R(\sigma^-), \quad (\partial_+\vec{x}_L)^2 = p_0^2 = (\partial_-\vec{x}_R)^2$$

$$\partial_+\vec{x}_L = p^0\left(\frac{2\mathbf{f}}{1+\mathbf{f}^2}, \frac{1-\mathbf{f}^2}{1+\mathbf{f}^2}\varepsilon_L\right), \quad \partial_-\vec{x}_R = p^0\left(\frac{2\mathbf{g}}{1+\mathbf{g}^2}, \frac{1-\mathbf{g}^2}{1+\mathbf{g}^2}\varepsilon_R\right)$$

(48)

---

[6] Note that the definition of fold in ref.[16] does not take into account that the map from the world sheet to spacetime may be many to one (i.e. a region maped to a segment), as explained in footnote 3. This feature may be important in the formulation of folds and their interactions in the path integral approach. In particular, the description of folds in the conformal gauge, as in the present paper, may eventually prove to be a more convenient mathematical formulation than the one used in [16].

where $\mathbf{f}(\sigma^+)$, $\mathbf{g}(\sigma^-)$ are arbitrary periodic vectors in $d-2$ dimensions, *which could be discontinuous*, and $\varepsilon_L(\sigma^+)$, $\varepsilon_R(\sigma^-)$ take the values ± 1 in patches of the corresponding variables such that the sign patterns repeat periodically (as in the 2D string). When $\mathbf{f}, \mathbf{g}$ are both zero the solution reduces to the 2 dimensional BBHP case. In general, the presence of discontinuous $\varepsilon_L, \varepsilon_R$, and the discontinuities in $\mathbf{f}(\sigma^+)$, $\mathbf{g}(\sigma^-)$ gives a larger set of solutions, which include strings that are partially or fully folded. Discontinuities are allowed since the differential equations are first order in the derivatives ∂_+ and ∂_-. Such solutions are usually missed in the lightcone gauge even in the flat classical theory (therefore, the lightcone "gauge" is not really a gauge).
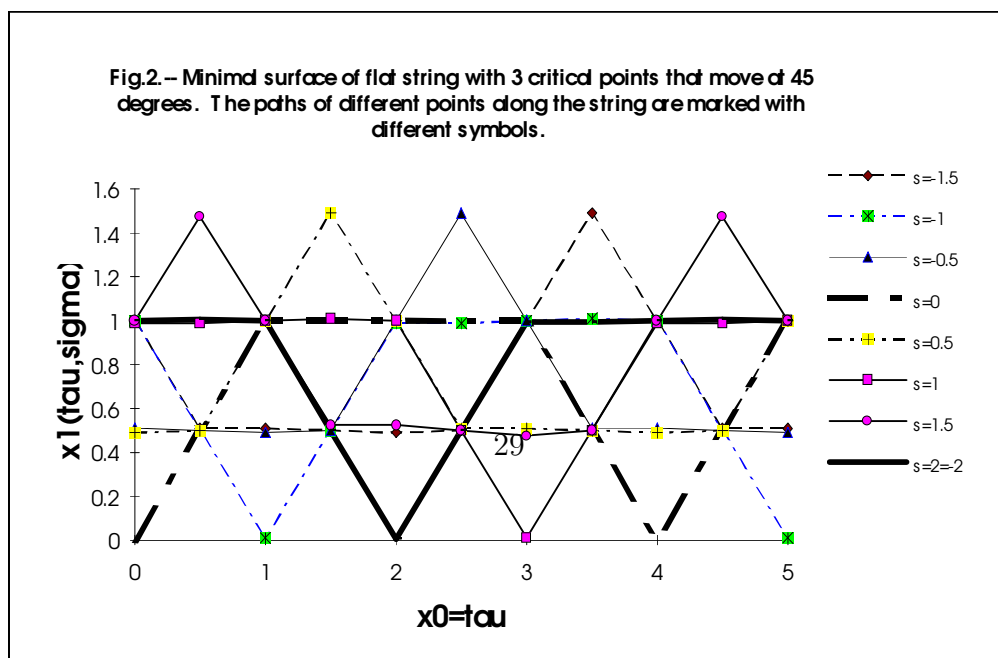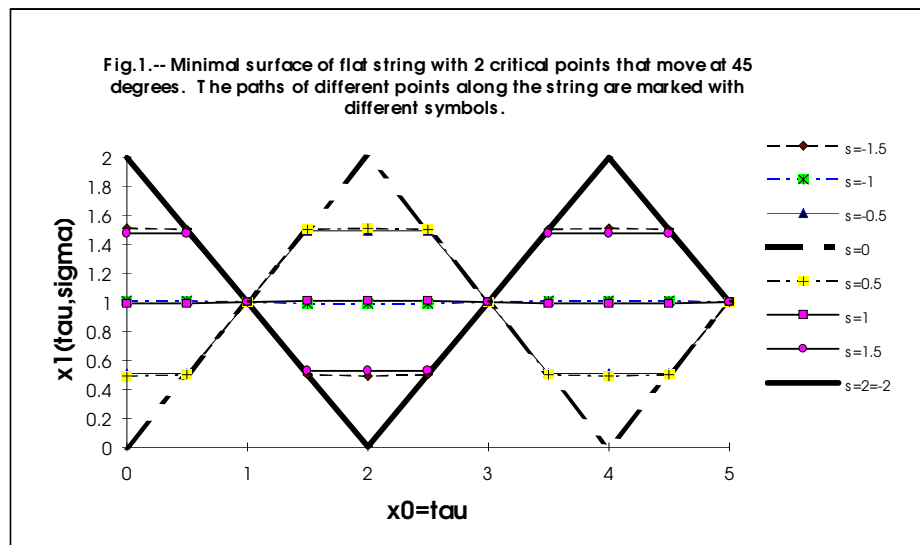
The curved space-time analogs of such solutions in higher dimensions are presently under investigation. We suspect that the inclusion of the quantum states corresponding to such solutions may lead to a consistent quantum theory in less than 26 dimensions. As already emphasized earlier in the paper, the free string is perfectly consistent as a quantum theory for $c < 26$, including the folded states. The interacting quantum string with folds remains as an open possibility.

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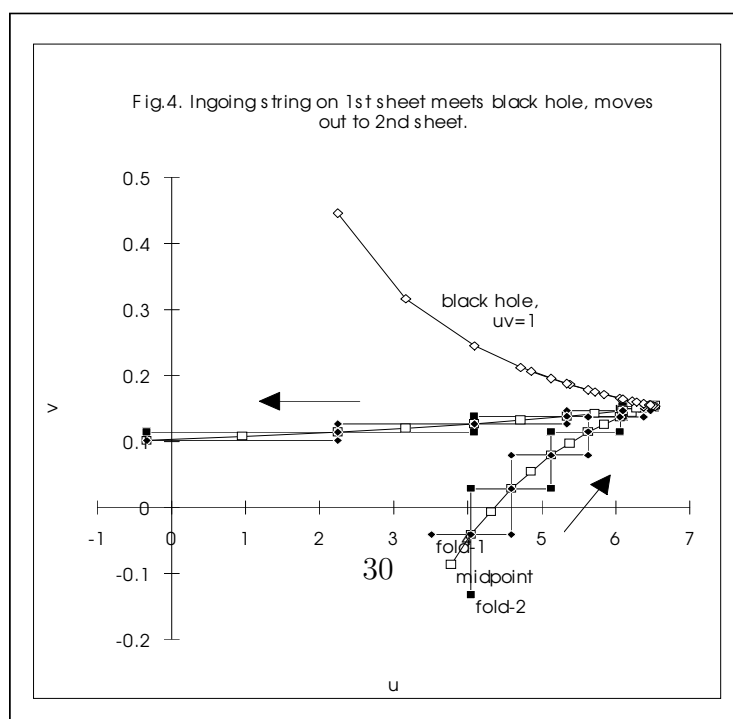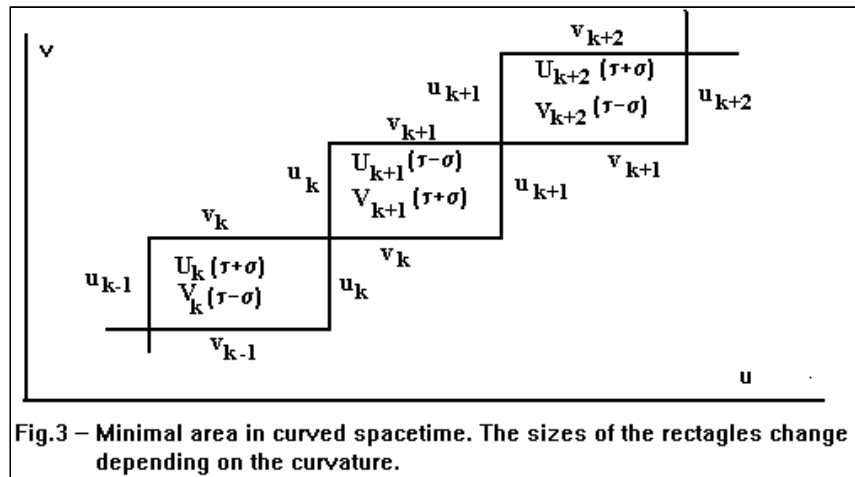[20] I. Bars and K. Sfetsos, Phys. Rev. **D46** (1992) 4495.

Figures

Fig.1.-- Minimal surface of flat string with 2 critical points that move at 45 degrees. The paths of different points along the string are marked with different symbols.

Fig.2.-- Minimal surface of flat string with 3 critical points that move at 45 degrees. The paths of different points along the string are marked with different symbols.

29

Fig.3 – Minimal area in curved spacetime. The sizes of the rectagles change depending on the curvature.

Fig.4. Ingoing string on 1st sheet meets black hole, moves out to 2nd sheet.

black hole,
uv=1

fold-1
midpoint
fold-2

30

Fig.5. String minimal area tunnels to forbidden region beyond black hole. Arrows along trajectories of midpoint.

(axis labels: v, u)

black hole, uv=1

fold-1

fold-2

midpoint

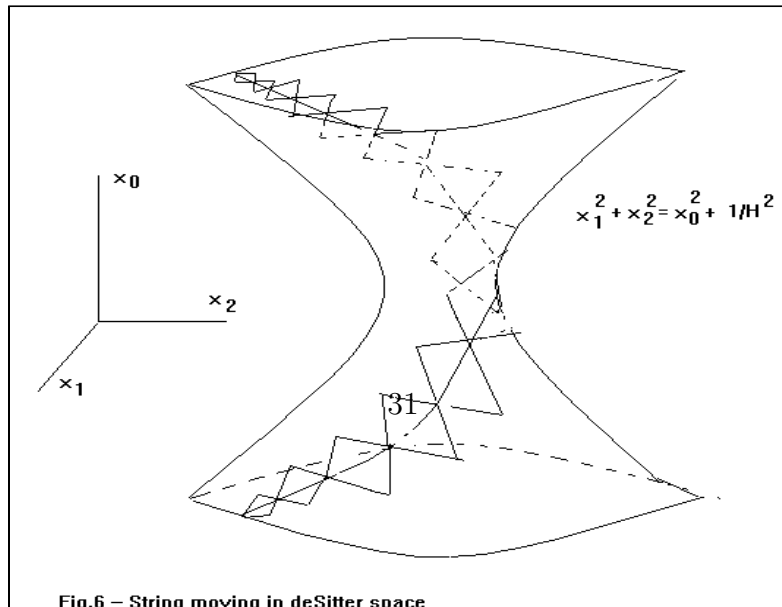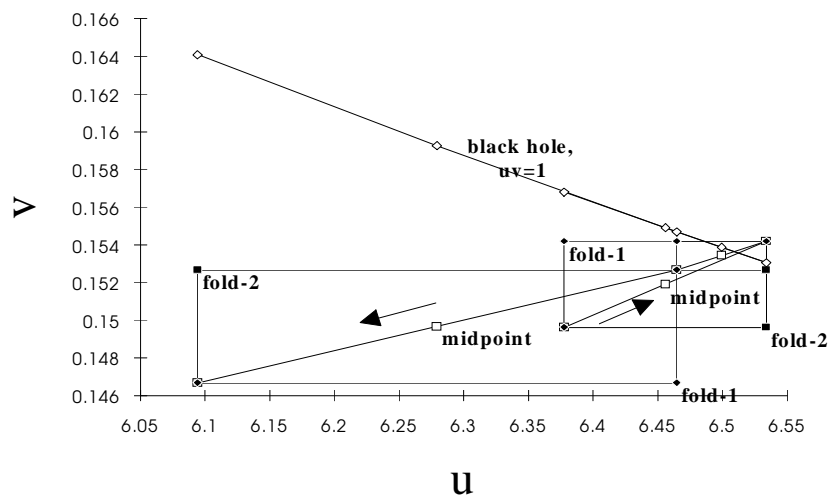midpoint

fold-2

fold+1

x_0

x_2

x_1

$x_1^2 + x_2^2 = x_0^2 + 1/H^2$

31

Fig.6 – String moving in deSitter space